that is to say, that any choice of material coordinate system is irrelevant to the physics of the problem while the material might exhibit different properties in different directions as the consequence of the dependence of the moduli upon stresses. Throughout the rest of this paper we make the following assumptions.

1) The elastic coefficients reduce to those of classical elastic materials in the absence of either compression or tension in the system, however those two sets of elastic coefficients corresponding to the case of no tension or no compression are not numerically the same.

2) The constants related to the coupling of stresses and stress signs are assumed to be negligible compared to those of direct effect that is

$$C_{ijk}^{\mp} = 0 \text{ if } J \neq k$$

By incorporating the aforementioned assumptions together with isotropy requirement for the cases of no tension or no compression and symmetry condition (14) and (15), one would obtain the following:

$$\begin{aligned} C_{ijk}^{\pm} &= k_1^{\pm} \text{ if } i = j = k \\ C_{ijk}^{\pm} &= k_2^{\pm} \text{ if } i \neq j \qquad j = k \\ C_{ijk}^{\pm} &= 0 \text{ if } J \neq k \end{aligned} \tag{16}$$

or in a more compact form

$$C_{ijk}^{\mp} = [(K_1^{\mp} - K_2^{\mp})\delta_{ij} + K_2^{\mp}]\delta_{jk} \tag{17}$$

the assumption that $K_1^{+} = K_2^{-}$, would further simplify the constitutive relations to

$$\varepsilon_\alpha = [1/E^+ U(\sigma_\alpha) + 1/E^-(-\sigma_\alpha)]\sigma_\alpha + C(\sigma_\beta + \sigma_\gamma) \tag{18a}$$

$$\varepsilon_\beta = [1/E^+ U(\sigma_\beta) + 1/E^- U(-\sigma_\beta)]\sigma_\beta + C(\sigma_\alpha + \sigma_\gamma) \tag{18b}$$

$$\varepsilon_\gamma = [1/E^+ U(\sigma_\gamma) + 1/E^- U(-\sigma_\gamma)]\sigma_\gamma + C(\sigma_\alpha + \sigma_\beta)$$
$$\alpha = \mathrm{I},\ \beta = \mathrm{II},\ \gamma = \mathrm{III} \tag{18c}$$

where

$$K_1^+ = 1/E^+ \quad K_1^- = 1/E^- \quad K_1^{\mp} = C$$

Equations (18) are equivalent to a single representation of those presented by Ambartsumyan. He deduced the equality of $K_2^+ = K_2^- = -v^-/E^- = -v^+/E^+$ from the symmetry of elasticity constants by incorrect use of strain energy function. It was shown here that the strain energy can be only used in a restricted sense and that does imply a special symmetry condition as given by Eqs. (14) and (15). In a previous paper,[9] it was shown that the matrix of elastic constants for anisotropic case can not be symmetric and such assumption would lead to contradictory results. The above assumption, however, for isotropic case presents no difficulty.

The preceding derivations are referred to principal axes of stress tensor. The equations relative to an arbitrary cartesian system can be obtained by use of well-known transformation relations.

V. Constitutive Equations for Orthotropic Bilinear Materials

As an example we may cite unidirectional reinforced composites. Let the cartesian coordinate system (x, y, z) coincide with the axes of material symmetry. By using the same line of approach as in the isotropic case, the stress-strain relations with respect to the principal directions of the stresses are obtained to be

$$\varepsilon_i = C_{ijk}^{+}\sigma_j U(\sigma_k) + C_{ijk}^{-}\sigma_j U(-\sigma_k) \qquad i,j,k = 1,\ldots,6$$
$$\sigma_j,\ \sigma_k = 0 \text{ for } j,k = 4,5,6 \tag{19}$$

where

$$\begin{aligned} C_{ijk}^{\mp} &= 0 \text{ if } j \neq k \\ C_{ijj}^{\mp} &= C_{jij}^{\mp} \end{aligned} \tag{20}$$

and

$$C_{ijj}^{\mp} = \sum_{m=1}^{6} \sum_{m=1}^{6} a_{mn}^{\mp} q_{mi} q_{mj}$$

where q_{mi} are given in terms of direction cosines of the coordinate transformation.[10] The transformation here is from coordinate of principal stresses to that of (x, y, z) the coordinate of material symmetry.

The constants $a_{mn}^{\mp}$ are 18 elastic constants describing the classical orthotropic properties of the material in the state of no tension and no compression. These two 9 sets of elastic constants are, obviously, to be found from experiments.

The constitutive equations for two-dimensional orthotropic case, together with the proper transformation equations are given in Ref. 9. The latter equations relate the constants of bilinear materials to those of classical elasticity ones. It also has been shown that the matrix of elastic constant is not symmetric for orthotropic case. Some interesting features of composite materials were explained and accounted for by the present theory, these constitutive equations are used to formulate a beam theory for bimodulus elastic materials.[11]

References

[1] Clark, S. K., "The Plane Elastic Characteristics of Cord-Rubber Laminates," *Textile Research Journal*, Vol. 33, No. 4, April, 1963.

[2] Ambartsumyan, S. A., "The Axisymmetric Problem of a Circular Cylindrical Shell Made of Material with Different Strength in Tension and Compression," *Izvestia. Mekhanika*, No. 4, 1965, pp. 77–85; English translation, FTD-HT-23-1055-67.

[3] Ambartsumyan, S. A. and Khachatryan, A. A., "Elasticity for Materials with Different Resistance to Tension and Compression" (in Russian), *Mekhanika Tverdogo Tela*, No. 2, 1966.

[4] Ambartsumyan, S. A., "Equations for a Plane Problem of the Multi-resisting or Multi-modulus Theory of Elasticity" (in Russian), *Isvestia An Armyaskoy SSR, Mekhanika*, No. 2, 1966.

[5] Ambartsumyan, S. A. and Khachatryan, A. A., "Some Problems in the Zero-Moment Theory of Shell Made of Materials with Different Moduli," *SSR, Doklady*, Vol. 43. No. 4, 1966, pp. 198–204; English translation FTD-HT-23-1091-67.

[6] Ambartsumyan, S. A. and Khachatryan, A. A., "A Multi-modulus Elasticity Theory," *Mekhanika Tverdogo Tela*, No. 6, pp. 64–67, 1966; English translation TTF-10, 936, NASA.

[7] Ambartsumyan, S. A. and Khachatryan, A. A., "Zero-Moment Theory for Shells Made of Materials with Diverse Tensile Strength and Compression Strength" (in Russian), AIAA Ref. No. A67-25569.

[8] Ambartsumyan, S. A., "Basic Equations and Relations in the Theory of Elasticity of Anisotropic Bodies with Different Moduli in Tension and Compression," *Mekhanika Tverdogo Tela*, No. 3, pp. 51–61, 1969; trans. 12, Jan. 1970, Aerospace Corp., El Segundo, Calif.

[9] Tabaddor, F., "Two-Dimensional Bi-linear Orthotropic Elastic Materials," *Journal of Composite Materials*, Vol. 3, 1969.

[10] Lekhnitskii, S. G., *Theory of Elasticity of Anisotropic Elastic Body*, Holden-Day, 1963.

[11] Tabaddor, F., "Analysis for Beams Made of Bi-modulus Orthotropic Elastic Materials," to be published.

Bending of a Simply Supported Plate Clamped about a Central Circular Hole

Roy C. Fedotoff* and John F. Carney III†
University of Connecticut, Storrs, Conn.

Introduction

EXACT, closed-form solutions to plate bending problems with irregular boundaries are not always available. The point-matching method yields numerical solutions to boundary value

Received July 8, 1971. The computations were performed in the Computer Center of the University of Connecticut. This facility is supported in part by the National Science Foundation Grant GJ-9.

* Graduate Assistant, Civil Engineering Department; now Consulting Engineer, Bowe, Walsh and Associates, Huntington, N.Y.

† Associate Professor, Civil Engineering Department.

problems (for which there is a known series solution to the governing differential equation) by satisfying exactly the governing differential equation and satisfying the boundary conditions at certain discrete points on the boundary.

The first known paper to appear in the literature using point matching was presented in 1934 by Slater[1] in which he dealt with electronic energy bonds in metals. Barta[2] was the first to apply the method to a plate bending problem in 1937, and in 1948, Thorne[3] considered a square plate. Fend et al.[4] employed the technique in their study of a temperature distribution problem (1950). In 1960, a paper by Conway[5] opened the door to widespread application of the method in solid mechanics. Recently, Leissa et al.[6–9] have contributed heavily to the advancement of the technique.

This Note presents a study of the bending of a simply-supported square plate fixed along a central circular hole and subjected to a uniform transverse load, as shown by Fig. 1.

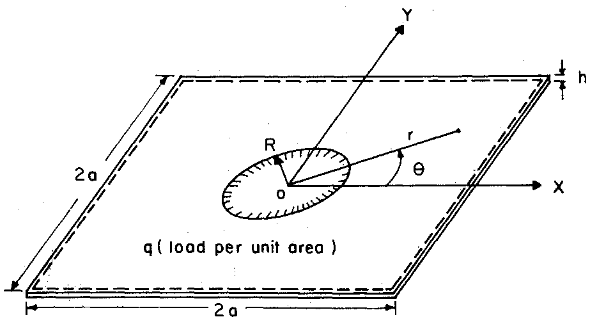

Fig. 1 Thin square plate fixed around central circular hole and simply supported along the outer edges and subjected to a uniformly distributed transverse load.

Analysis

The governing differential equation for the system is

$$\nabla^4 w = q/D \tag{1}$$

in which ∇^4 is the biharmonic operator, w is the middle surface deflection, D is the flexural rigidity of the plate, and q is the intensity of the distributed loading. The solution to the preceding equation can be written in the form

$$w = qr^4/64D + A_o + B_o \ln r + C_o r^2 + D_o r^2 \ln r + \sum_{n=4,8,12\ldots}^{\infty} (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) \cos n\theta \tag{2}$$

in which A_o, B_o, C_o, D_o, A_n, B_n, C_n, and D_n are undetermined coefficients.

The coefficients of Eq. (2) are determined by imposing the various boundary conditions at the inner and outer edges. A polar coordinate system lends itself to exactly satisfying the boundary conditions everywhere along the central circular hole (at $r = R$). However, the rectilinear edges of the plate do not conform to the polar coordinate system. These outer boundary conditions can at best be satisfied only in an approximate fashion.

The boundary conditions for the plate are

$$(w)_{r=R} = 0, \quad (\partial w/\partial r)_{r=R} = 0, \quad (w)_{\substack{\text{outside} \\ \text{boundary}}} = 0,$$

$$(M_n)_{\substack{\text{outside} \\ \text{boundary}}} = 0 \tag{3}$$

where M_n represents the bending moment in the direction normal to the edge of the plate.

Applying the inside boundary conditions to the solution yields

$$A_o = C_o R^2 (2 \ln R - 1) + D_o R^2 \ln R (2 \ln R) + (qR^4/64D)(4 \ln R - 1) \tag{4}$$

$$B_o = -(qR^4/16D) - 2C_o R^2 - D_o R^2 (1 + 2 \ln R) \tag{5}$$

$$B_n = (n-1) A_n R^{2n} + n C_n R^{2n+2} \tag{6}$$

and

$$D_n = -n A_n R^{2n-2} - (n+1) C_n R^{2n} \tag{7}$$

Substituting Eqs. (4–7) into Eq. (2) results in a deflection function containing $2n + 2$ unknown constants to be determined from the outer edge boundary conditions. It is clear that these outer edge boundary conditions cannot be satisfied exactly everywhere because of the shape of the outside boundary. A modified point matching method is employed to satisfy the outside boundary conditions in the least squares sense.

Because of the symmetry of the problem, only one-eighth of the plate need be considered. Truncating the series in Eq. (2) at $n = 24$ as per Leissa[9] leads to, on the boundary,

$$\begin{aligned}(w)_{\substack{x=0 \\ 0\le y\le a}} &= (q/64D)[R^4(4\ln R - 1) - 4R^4 \ln r + r^4] \\ &\quad + C_o[R^2(2\ln R - 1) - 2R^2 \ln r + r^2] \\ &\quad + D_o[R^2 \ln R(2\ln R) - R^2(1 + 2\ln R)\ln r + r^2 \ln r] \\ &\quad + \sum_{n=4,8,\ldots 24} \{A_n[r^n + (n-1)R^{2n} r^{-n} - nR^{(2n-2)} r^{-(n-2)}] \\ &\quad + C_n[^{n+2} + nR^{2n+2} r^{-n} - (n+1)R^{2n} r^{-(n-2)}]\cos n\theta]\} \\ &= 0 \end{aligned} \tag{8}$$

$$\begin{aligned}(M_n)_{\substack{x=a \\ 0\le y\le a}} &= (q/16)[2r^2\{1 + v + ([1-v]/2)\cos 2\phi\} \\ &\quad + R^4 r^{-2}(1-v)\cos 2\phi] - D\{2C_o[R^2 r^{-2} \\ &\quad \times (1-v)\cos 2\phi + 2(1+v)] + D_o[R^2 r^{-2}(1 + 2\ln R) \\ &\quad \times (1-v)\cos 2\phi + 2(1 + \ln r)(1+v) + (1-v)\cos 2\phi] \\ &\quad + \sum_{n=4,8,\ldots 24} \{A_n[(n)(n-1)r^{n-2}(1-v)\cos(n\theta + 2\phi) \\ &\quad + n(n-1)(n+1)R^{2n} r^{-(n+2)}(1-v)\cos(n\theta - 2\phi) \\ &\quad - 2n(n-1)R^{(2n-2)} r^{-n}[-(1+v)\cos n\theta \\ &\quad + (n/2)(1-v)\cos(n\theta - 2\phi]\} \\ &\quad + C_n\{n^2(n+1)R^{(2n+2)} r^{-(n+2)}(1-v)\cos(n\theta - 2\phi) \\ &\quad + (2n+2)r^n[(1+v)\cos n\theta + (n/2)(1-v)\cos(n\theta + 2\phi)]\}\} \\ &= 0 \end{aligned} \tag{9}$$

in which ϕ is the angle between the normal to the plate and the radial direction, r.

There are fourteen unknown coefficients in Eq. (8), and these same coefficients appear in Eq. (9). Seven points on one-eighth of the outside boundary could therefore be selected and the fourteen unknowns determined. Instead of adopting this approach, fourteen boundary points are selected and the boundary conditions satisfied in the least squares sense. The advantage in doing this lies in the fact that, since the boundary conditions are satisfied in the least squares sense at fourteen points (instead of exactly satisfying them at seven points), the stress resultant expressions involving derivatives of w are more accurate in the vicinity of the boundary. Note that, in applying this modified point matching technique, the plate equation has been satisfied exactly as has the inside boundary conditions.

Results

Some typical results are presented in Figs. 2–4. In all cases, a Poisson's ratio of 0.30 is employed. Dimensionless deflections and distances ($\rho = r/a$) are shown in Fig. 2 for three values of θ when $R/a = \frac{1}{6}$. The locations of maximum deflection are seen to move toward the hole with increasing θ and the magnitude of these maxima also increase with increasing θ. It is interesting to note that the slope at the outer edge decreases as θ approaches $\pi/4$, as would be expected. In Fig. 3, radial bending moment stress resultants are plotted for various R/a ratios at $\theta = 0$. Figure 4 shows that the tangential bending moment stress resultants along the hole diminish by roughly a factor of two for each added increment in the R/a ratio.

To determine the accuracy of the results obtained, an error analysis was performed. First, a residue was calculated to see how well the boundary conditions (w and M) were satisfied at each of the fourteen points along the outer boundary. Then a measure of the maximum value of the error was found for the calculated deflections and bending moment stress resultants by taking the ratio of the value of deflection or radial bending moment stress resultants at the outer edge on $\theta = 0$, to their respective maximum values within the plate.

The deflection and bending moment residues were of the order of 10^{-13}, while the values of the error ratios were of the order of 10^{-4}, indicating a high degree of accuracy.

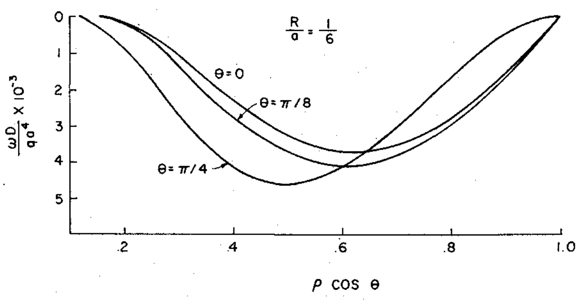

Fig. 2 Values of dimensionless deflection function, w, vs dimensionless radius, $\rho \cos\theta$, for $R/a = \frac{1}{6}$, $\nu = 0.30$, and $\theta = 0, \pi/8$ and $\pi/4$.

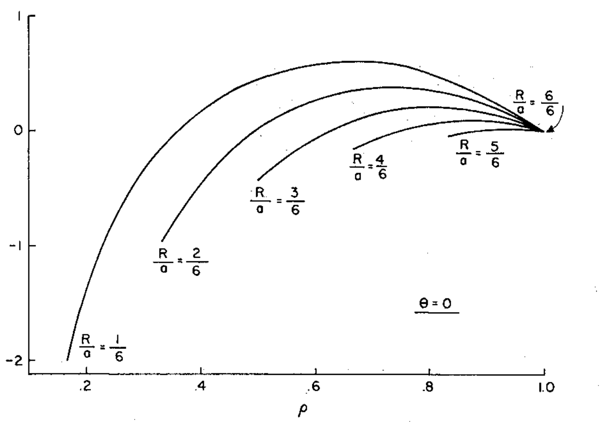

Fig. 3 Values of dimensionless radial bending moment, M_r, vs dimensionless radius, ρ, for $\theta = 0$, $\nu = 0.30$, and $R/a = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$, and $\frac{6}{6}$.

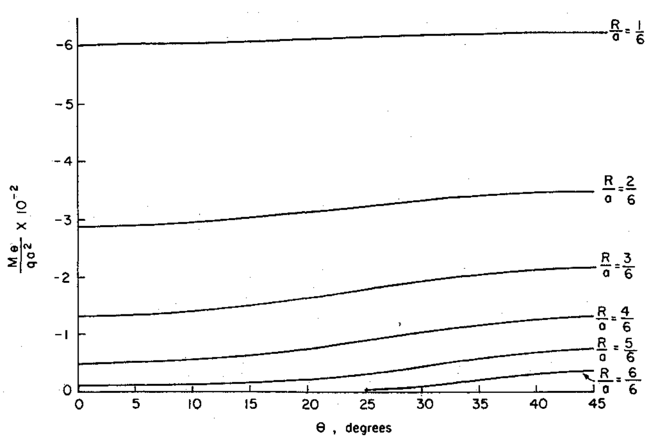

Fig. 4 Values of dimensionless tangential bending moment, M_θ, along the circular hole, for $\nu = 0.30$ and $R/a = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$, and $\frac{6}{6}$.

References

[1] Slater, J. C., "Electron Energy Bands in Metals," *Physical Review*, Vol. 45, 1934, pp. 794–801.

[2] Barta, J., "On the Numerical Solution of a Two-Dimensional Elasticity Problem," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 7, 1937, pp. 184–185.

[3] Thorne, C. J., "Square Plates Fixed at Points," *Journal of Applied Mechanics*, Vol. 15, 1948, pp. 73–79.

[4] Fend, F. A., Baroody, E. M., and Bell, J. C., "An Approximate Calculation of the Temperature Distribution Surrounding Coolant Holes in a Heat-Generating Solid," BMI Rept. T-42, 1950, Battelle Memorial Inst., Columbus, Ohio.

[5] Conway, H. D., "The Approximate Analysis of Certain Boundary Value Problems," *Journal of Applied Mechanics*, Vol. 27E, 1960, pp. 275–277.

[6] Niedenfuhr, F. W., Leissa, A. W., and Lo, C. C., "A Study of the Point Matching Method as Applied to Thermally and Transversely Loaded Plates and other Boundary Value Problems," TR AFFDL-TR-64-159, 1964, The Ohio State Univ. Research Foundation. Columbus, Ohio.

[7] Lo, C. C., Niedenfuhr, F. W., and Leissa, A. W., "Further Studies in the Application of the Piont Matching Technique to Plate Bending and Other Harmonic and Biharmonic Boundary Value Problems," TR AFFDL-TR-65-114, 1966, The Ohio State Univ. Research Foundation, Columbus, Ohio.

[8] Hopper, A. T., Leissa, A. W., Hulbert, L. E., and Clausen, W. E., "Numerical Analysis of Equilibrium and Eigenvalue Problems," TR AFFDL-TR-67-121, 1967, Battelle Memorial Institute, Columbus, Ohio.

[9] Lo, C. C. and Leissa, W. E., "Bending of Plates with Circular Holes," *Acta Mechanica*, Vol. 1, 1967, pp. 64–76.

Nonlinear Filtering for Random Signals in Statistically Unknown Noise

JAMES T. LO*
Harvard University, Cambridge, Mass.

Introduction and Problem Formulation

IN many problems of satellite orbit determination, aircraft navigation, and missile tracking, the environment of the sensor keeps changing from time to time. It is quite impractical, sometimes impossible to collect the statistical data of the noise incurred in the observation. The purpose of this Note is to present a natural and effective formulation of the concerned filtering problem and to deduce computable filtering equations.

The state evolution of moving objects under observation is assumed to be continuous in time. It is under disturbances and hence is nondeterministic. We assume it has strong Markov property. With all these properties, a natural mathematical model to use is the following vector stochastic differential equation in the sense of K. Ito,[4,5]

$$dx_t = g(x_t, t)dt + \sigma(x_t, t)d\beta_t \tag{1}$$

$$x_0 = b \tag{2}$$

where β_t is a standard Brownian motion and b a random variable.

Suppose the process x_t is observed in a noisy manner on the time interval $(0, T)$ by

$$y_t = h(x_t, t) + v_t \tag{3}$$

where v_t is the observational noise of which the statistics are not known to us, but it is assumed that v_t is of second order and is continuous in quadratic mean.

Now we want to estimate the present state x_s based on all the past observation $0_s = \{y_t, t\varepsilon[0, s]\}$ in some optimal fashion. Let

Received November 12, 1971; revision received November 29, 1971. This work was supported in part by Air Force Office of Scientific Research under Grants AFOSR-1244-67B, NASA Grant NGL05-020-073, and Army Contract DA-31-124-ARO(D)394.

Index category: Navigation, Control, and Guidance Theory.

* Research Fellow in Applied Mathematics, Department of Engineering and Applied Physics.